

Discrepancy estimates for index-transformed uniformly distributed sequences

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Abstract

In this paper we show discrepancy bounds for index-transformed uniformly distributed sequences. From a general result we deduce very tight lower and upper bounds on the discrepancy of index-transformed van der Corput-, Halton-, and (t, s) -sequences indexed by the sum-of-digits function. We also analyze the discrepancy of sequences indexed by other functions, such as, e.g., $\lfloor n^\alpha \rfloor$ with $0 < \alpha < 1$.

Keywords: Discrepancy, uniform distribution, van der Corput-sequence, Halton-sequence, (t, s) -sequence, sum-of-digits function.

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1 Introduction

A sequence $(\mathbf{y}_n)_{n \geq 0}$ in the unit-cube $[0, 1]^s$ is said to be *uniformly distributed modulo one* if for all intervals $[\mathbf{a}, \mathbf{b}] \subseteq [0, 1]^s$ it is true that

$$\lim_{N \rightarrow \infty} \frac{\#\{n : 0 \leq n < N, \mathbf{y}_n \in [\mathbf{a}, \mathbf{b}]\}}{N} = \text{vol}([\mathbf{a}, \mathbf{b}]). \quad (1)$$

A quantitative version of (1) can be stated in terms of discrepancy. For an infinite sequence $(\mathbf{y}_n)_{n \geq 0}$ in $[0, 1]^s$ its *discrepancy* is defined as

$$D_N((\mathbf{y}_n)_{n \geq 0}) := \sup_{[\mathbf{a}, \mathbf{b}] \subseteq [0, 1]^s} \left| \frac{\#\{n : 0 \leq n < N, \mathbf{y}_n \in [\mathbf{a}, \mathbf{b}]\}}{N} - \text{vol}([\mathbf{a}, \mathbf{b}]) \right|,$$

where the supremum is extended over all sub-intervals $[\mathbf{a}, \mathbf{b}]$ of $[0, 1]^s$. For a given finite sequence $X = (\mathbf{x}_1, \dots, \mathbf{x}_M)$ we write $D_M(X)$ for the discrepancy of X with the obvious adaptations in the above definition. An infinite sequence is uniformly distributed modulo one if and only if its discrepancy tends to zero as N goes to infinity. However, convergence

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of the discrepancy to zero cannot take place arbitrarily fast. It follows from a result of Roth [28] that for any infinite sequence $(\mathbf{y}_n)_{n \geq 0}$ in $[0, 1]^s$ we have $ND_N((\mathbf{y}_n)_{n \geq 0}) \geq c_s(\log N)^{s/2}$ for infinitely many values of $N \in \mathbb{N}$ (by \mathbb{N} we denote the set of positive integers, and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). An improvement of this bound can be obtained from [4]. For the special case $s = 1$, Schmidt [29] (see also [2]) showed that for any infinite sequence $(y_n)_{n \geq 0}$ in $[0, 1)$ we have $ND_N((y_n)_{n \geq 0}) \geq \frac{\log N}{66 \log 4}$ for infinitely many values of $N \in \mathbb{N}$. This result is best possible with respect to the order of magnitude in N . An excellent introduction to this topic can be found in the book of Kuipers and Niederreiter [20] (see also [6, 9, 21, 24]).

Well known examples of uniformly distributed sequences are $(n\alpha)$ -sequences (also called Kronecker-sequences, see [9, 20]), van der Corput-sequences and their multivariate analogues called Halton-sequences (see [6, 19, 20, 24]), as well as (digital) (t, s) -sequences (see [6, 24]).

In recent years, also the distribution properties of index-transformed uniformly distributed sequences have been studied, especially for the examples mentioned above. In this paper, we mean by an index-transformed sequence of a sequence $(x_n)_{n \geq 0}$ a sequence $(x_{f(n)})_{n \geq 0}$, where $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$. Note that $(x_{f(n)})_{n \geq 0}$ is in general no subsequence of $(x_n)_{n \geq 0}$ since we do *not* require that f is strictly increasing.

For instance, the distribution properties of index-transformed Kronecker-sequences indexed by the sum-of-digits function were studied in [5, 8, 30, 31]. For this special case, very precise results can be found in [8]. In [7] the well-distribution of index-transformed Kronecker-sequences indexed by q -additive functions is considered. Furthermore, in [26] a discrepancy bound for van der Corput-sequences in bases of the form $b = 5^\ell$, $\ell \in \mathbb{N}$, indexed by Fibonacci numbers is shown. The papers [17, 18, 26] deal with index-transformed van der Corput-, Halton-, and (t, s) -sequences.

In this paper we are specifically interested in discrepancy bounds for sequences indexed by the q -ary sum-of-digits function and related functions and, furthermore, for sequences indexed by “moderately” monotonically increasing sequences, as for example $\lfloor n^\alpha \rfloor$ with $0 < \alpha < 1$. For an integer $q \geq 2$ and $n \in \mathbb{N}_0$ with base q expansion $n = r_0 + r_1q + r_2q^2 + \dots$ the q -ary *sum-of-digits function* is defined by $s_q(n) := r_0 + r_1 + r_2 + \dots$.

Previously, it has been shown in [18] that the sequence $(\mathbf{x}_{s_q(n)})_{n \geq 0}$, indexed by the q -ary sum-of-digits function, where $(\mathbf{x}_n)_{n \geq 0}$ denotes the Halton-sequence in co-prime bases b_1, \dots, b_s is uniformly distributed modulo one. The proof of this result is due to the fact that the sequence generated by the q -ary sum-of-digits function is uniformly distributed in \mathbb{Z} , see, for example, [12, 27]. In this paper we provide very tight lower and upper bounds on the discrepancy of index-transformed van der Corput-, Halton-, and (t, s) -sequences indexed by the sum-of-digits function.

This paper is structured as follows. In Section 2, we provide basic definitions and notation used throughout the subsequent sections. In Section 3, we prove a general theorem (Theorem 1) which will be of great importance in discussing sequences indexed by the sum-of-digits function. In Section 4 we present a concrete application of Theorem 1 which leads to the aforementioned tight bounds on the discrepancy of Halton- and (t, s) -sequences indexed by $s_q(n)$. Furthermore, we discuss a refinement of these results for van der Corput-sequences. Finally, in Section 5, we deal with discrepancy bounds for

sequences which are obtained by certain moderately increasing index sequences, such as, e.g., $\lfloor n^\alpha \rfloor$ with $0 < \alpha < 1$.

2 Notation and basic definitions

We first outline the definitions of the sequences studied in this paper, namely van der Corput-, Halton-, and (t, s) -sequences.

Let $b \geq 2$ be an integer. A *van der Corput-sequence* $(x_n)_{n \geq 0}$ in base b is defined by $x_n = \varphi_b(n)$, where for $n \in \mathbb{N}_0$, with base b expansion $n = a_0 + a_1b + a_2b^2 + \dots$, the so-called *radical inverse function* $\varphi_b : \mathbb{N}_0 \rightarrow [0, 1)$ is defined by

$$\varphi_b(n) := \frac{a_0}{b} + \frac{a_1}{b^2} + \frac{a_2}{b^3} + \dots.$$

It is well known that for any base $b \geq 2$ the corresponding van der Corput-sequence is uniformly distributed modulo one and that $ND_N((x_n)_{n \geq 0}) = O(\log N)$, see, for example, [3, 6, 20].

If we choose co-prime integers $b_1, \dots, b_s \geq 2$, then s one-dimensional van der Corput-sequences can be combined to an s -dimensional uniformly distributed sequence with points $\mathbf{x}_n := (\varphi_{b_1}(n), \dots, \varphi_{b_s}(n))$ for $n \in \mathbb{N}_0$. This sequence is called a *Halton-sequence* and it is known that its discrepancy is of order $(\log N)^s/N$, see [1, 6, 10, 11, 13, 19, 22, 24]. Note that Halton-sequences are a direct generalization of van der Corput-sequences, so van der Corput-sequences can be viewed as one-dimensional Halton-sequences, and indeed Halton-sequences are sometimes also referred to as van der Corput-Halton-sequences (see, e.g., [20]). However, as there will be results in this paper which only hold for the one-dimensional case, it will be useful to explicitly distinguish van der Corput-sequences (which we use for the one-dimensional variant) from Halton-sequences (which we use for the multi-dimensional variant).

Another type of sequences we will be concerned with in this paper are (t, s) -sequences, for the definition of which we need the definition of elementary intervals and (t, m, s) -nets in base b .

For an integer $b \geq 2$, an *elementary interval* in base b is an interval of the form $\prod_{i=1}^s [a_i b^{-d_i}, (a_i + 1)b^{-d_i}) \subseteq [0, 1)^s$, where a_i, d_i are non-negative integers with $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$.

Let t, m , with $0 \leq t \leq m$, be integers. Then a (t, m, s) -net in base b is a point set $(\mathbf{y}_n)_{n=0}^{b^m-1}$ in $[0, 1)^s$ such that any elementary interval in base b of volume b^{t-m} contains exactly b^t of the \mathbf{y}_n .

Furthermore, we call an infinite sequence $(\mathbf{x}_n)_{n \geq 0}$ a (t, s) -sequence in base b if the subsequence $(\mathbf{x}_n)_{n=kb^m}^{(k+1)b^m-1}$ is a (t, m, s) -net in base b for all integers $k \geq 0$ and $m \geq t$. It is known (see, e.g., [6, 23, 24]) that a (t, s) -sequence is particularly evenly distributed if the value of t is small. In particular, it can be shown that the discrepancy of a (t, s) -sequence in base b is of order $b^t(\log N)^s/N$, see, e.g., [6, 23, 24].

A very important sub-class of (t, s) -sequences is that of digital (t, s) -sequences, which are defined over algebraic structures like finite fields or rings. For the sake of simplicity,

we restrict ourselves to digital sequences over finite fields \mathbb{F}_p of prime order p . Again for the sake of simplicity we do not distinguish, here and later on, between elements in \mathbb{F}_p and the set of integers $\{0, 1, \dots, p-1\}$ (equipped with arithmetic operations modulo p).

For a vector $\mathbf{c} = (c_1, c_2, \dots) \in \mathbb{F}_p^\infty$ and for $m \in \mathbb{N}$ we denote the vector in \mathbb{F}_p^m consisting of the first m components of \mathbf{c} by $\mathbf{c}(m)$, i.e., $\mathbf{c}(m) = (c_1, \dots, c_m)$. Moreover, for an $\mathbb{N} \times \mathbb{N}$ matrix C over \mathbb{F}_p and for $m \in \mathbb{N}$ we denote by $C(m)$ the left upper $m \times m$ submatrix of C .

For $s \in \mathbb{N}$ and $t \in \mathbb{N}_0$, choose $\mathbb{N} \times \mathbb{N}$ matrices C_1, \dots, C_s over \mathbb{F}_p with the following property. For every $m \in \mathbb{N}$, $m \geq t$, and all $d_1, \dots, d_s \in \mathbb{N}_0$ with $d_1 + \dots + d_s = m - t$, the vectors

$$\mathbf{c}_1^{(1)}(m), \dots, \mathbf{c}_{d_1}^{(1)}(m), \dots, \mathbf{c}_1^{(s)}(m), \dots, \mathbf{c}_{d_s}^{(s)}(m)$$

are linearly independent in \mathbb{F}_p^m . Here $\mathbf{c}_i^{(j)}$ is the i -th row vector of the matrix C_j .

For $n \in \mathbb{N}_0$ let $n = n_0 + n_1p + n_2p^2 + \dots$ be the base p representation of n . For every index $1 \leq j \leq s$ multiply the digit vector $\mathbf{n} = (n_0, n_1, \dots)^\top$ by the matrix C_j ,

$$C_j \cdot \mathbf{n} =: (x_{n,j}(1), x_{n,j}(2), \dots)^\top$$

(note that the matrix-vector multiplication is performed over \mathbb{F}_p), and set

$$x_n^{(j)} := \frac{x_{n,j}(1)}{p} + \frac{x_{n,j}(2)}{p^2} + \dots$$

Finally set $\mathbf{x}_n := (x_n^{(1)}, \dots, x_n^{(s)})$. A sequence $(\mathbf{x}_n)_{n \geq 0}$ constructed in this way is called a *digital (t, s) -sequence over \mathbb{F}_p* . The matrices C_1, \dots, C_s are called the *generator matrices* of the sequence.

To guarantee that the points \mathbf{x}_n lie in $[0, 1)^s$ (and not just in $[0, 1]^s$) we assume that for each $1 \leq j \leq s$ and $w \geq 0$ we have $c_{v,w}^{(j)} = 0$ for all sufficiently large v , where $c_{v,w}^{(j)}$ are the entries of the matrix C_j (see [24, p.72, condition (S6)] for more information).

Throughout the paper we use the following notation. For functions $f, g : \mathbb{N} \rightarrow \mathbb{R}$, where $f \geq 0$, we write $g(n) = O(f(n))$ or $g(n) \ll f(n)$, if there exists a $C > 0$ such that $|g(n)| \leq Cf(n)$ for all sufficiently large $n \in \mathbb{N}$. If we would like to stress that the quantity C may also depend on other variables than n , say $\alpha_1, \dots, \alpha_w$, which will be indicated by writing $\ll_{\alpha_1, \dots, \alpha_w}$.

3 A general theorem

In this section we present a general result for the discrepancy of sequences of the form $(\mathbf{x}_{g(n)})_{n \geq 0}$, for a particular class of functions $g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$. Here and in the following, a sequence $(a_k)_{k \in \mathbb{N}_0}$ is called *unimodal* if the sequence $(a_{k+1} - a_k)_{k \in \mathbb{N}_0}$ has exactly one change of sign.

Furthermore, we need the concept of the so-called *uniform discrepancy* of a sequence. The uniform discrepancy of a sequence $(\mathbf{x}_n)_{n \geq 0}$ in $[0, 1)^s$ is defined as

$$\tilde{D}_N((\mathbf{x}_n)_{n \geq 0}) := \sup_{k \in \mathbb{N}_0} D_N((\mathbf{x}_{n+k})_{n \geq 0}).$$

Theorem 1. Let $(\mathbf{x}_n)_{n \geq 0}$ be an s -dimensional sequence with uniform discrepancy $\tilde{D}_N = \tilde{D}_N((\mathbf{x}_n)_{n \geq 0})$, and let $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ be a non-decreasing function such that $N\tilde{D}_N \leq f(N)$ for $N \in \mathbb{N}_0$.

Let $g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$. Furthermore, let $(N_j)_{j \geq 0}$ be a strictly increasing sequence in \mathbb{N} with $1 = N_0$, and assume that $(N_j)_{j \geq 0}$ is a divisibility chain, i.e., $N_0 | N_1$, $N_1 | N_2$, $N_2 | N_3$, etc. Define, for $k \in \mathbb{N}_0$,

$$G_{A,j}(k) := \#\{n : AN_j \leq n < (A+1)N_j, g(n) = k\}.$$

Then the following two assertions hold.

1. For $N \in \mathbb{N}$ with $N_d \leq N < N_{d+1}$ we have $ND_N((\mathbf{x}_{g(n)})_{n \geq 0}) \geq \max_{k \in \mathbb{N}_0} G_{0,d}(k)$.
2. Assume that $G_{A,j}(k)$ is unimodal in k for all $j \in \mathbb{N}_0$ and all $A \in \mathbb{N}_0$, and put

$$G_j := \max_{k, A \in \mathbb{N}_0} G_{A,j}(k) \text{ for } j \in \mathbb{N}_0.$$

For $j \in \mathbb{N}_0$ and $A \in \mathbb{N}_0$ let

$$v_{A,j} := \#\{k \in \mathbb{N}_0 : g(n) = k \text{ for } AN_j \leq n < (A+1)N_j\}$$

and put

$$v_j := \max_{A \in \mathbb{N}_0} v_{A,j}.$$

Then for $N \in \mathbb{N}$ with $N_d \leq N < N_{d+1}$ we have

$$ND_N((\mathbf{x}_{g(n)})_{n \geq 0}) \leq \sum_{j=0}^d \frac{N_{j+1}}{N_j} G_j f(v_j).$$

Proof. 1. To show the lower bound choose a non-negative integer κ such that $\tilde{G}_d = G_{0,d}(\kappa) = \max_{k \in \mathbb{N}_0} G_{0,d}(k)$. Then the number of $n \in \{0, \dots, N-1\}$ such that $\mathbf{x}_{g(n)} = \mathbf{x}_\kappa$ is at least \tilde{G}_d and hence, with an arbitrarily small interval containing \mathbf{x}_κ we obtain

$$D_N((\mathbf{x}_{g(n)})_{n \geq 0}) \geq \frac{\tilde{G}_d}{N}.$$

2. To prove the upper bound let

$$N = a_d N_d + a_{d-1} N_{d-1} + \dots + a_0 N_0,$$

with $a_j \in \mathbb{N}_0$ and

$$a_j \leq \frac{N_{j+1}}{N_j}; \text{ for } j \in \{0, \dots, d\}.$$

For $j \in \{0, \dots, d\}$ and $\ell \in \{0, \dots, a_j - 1\}$ we consider the sequence

$$X_{j,\ell} := (\mathbf{x}_{g(AN_j+k)})_{k=0}^{N_j-1}$$

where $AN_j := a_d N_d + \dots + a_{j+1} N_{j+1} + \ell N_j$ (strictly speaking, $A = A(j, \ell)$).

Since $G_{A,j}$ is unimodal we may assume that for $AN_j \leq n < (A+1)N_j$ the function $g(n)$ attains the values

$$w, w+1, \dots, w+v,$$

for some $w \in \mathbb{N}_0$ and some integer $v = v_{A,j} \leq v(j)$

Assume that the value $w+u_1$ with $0 \leq u_1 \leq v$ is attained most often, the value $w+u_2$ with $0 \leq u_2 \leq v$ is attained second most often, etc. \dots , and $w+u_v$ with $0 \leq u_v \leq v$ (indeed, $u_v \in \{0, v\}$) is attained least often. If $w+u_r$ and $w+u_{r+1}$ are both attained the same number of times, then the order of them is of no relevance.

If we consider the sequence $X_{j,\ell}$ as a multi-set (i.e., multiplicity of the elements is relevant, but their order is not), then we can decompose $X_{j,\ell}$ into

$$\begin{array}{ll} G_{A,j}(w+u_1) - G_{A,j}(w+u_2) & \text{times } \{\mathbf{x}_{w+u_1}\} \\ G_{A,j}(w+u_2) - G_{A,j}(w+u_3) & \text{times } \{\mathbf{x}_{w+u_1}, \mathbf{x}_{w+u_2}\} \\ G_{A,j}(w+u_3) - G_{A,j}(w+u_4) & \text{times } \{\mathbf{x}_{w+u_1}, \mathbf{x}_{w+u_2}, \mathbf{x}_{w+u_3}\} \\ \dots & \\ G_{A,j}(w+u_{v-1}) - G_{A,j}(w+u_v) & \text{times } \{\mathbf{x}_{w+u_1}, \mathbf{x}_{w+u_2}, \dots, \mathbf{x}_{w+u_{v-1}}\} \\ G_{A,j}(w+u_v) - G_{A,j}(w+u_{v+1}) & \text{times } \{\mathbf{x}_{w+u_1}, \mathbf{x}_{w+u_2}, \dots, \mathbf{x}_{w+u_v}\}, \end{array}$$

where we formally set $G_{A,j}(w+u_{v+1}) := 0$. Note that because of the unimodality of $G_{A,j}(k)$, for $r \in \{1, \dots, v\}$, the sequence $\mathbf{x}_{w+u_1}, \mathbf{x}_{w+u_2}, \dots, \mathbf{x}_{w+u_r}$ is a sequence of the form $\mathbf{x}_B, \dots, \mathbf{x}_{B+r-1}$ for some B .

Then, using the assumptions of the theorem and the triangle inequality for the discrepancy (see [20, p. 115, Theorem 2.6]), we obtain

$$\begin{aligned} N_j D_{N_j}(X_{j,\ell}) &\leq \\ &\leq \sum_{r=1}^v (G_{A,j}(w+u_r) - G_{A,j}(w+u_{r+1})) r D_r(\{\mathbf{x}_{w+u_1}, \mathbf{x}_{w+u_2}, \dots, \mathbf{x}_{w+u_r}\}) \\ &\leq G_{A,j}(w+u_1) f(v_{A,j}) \\ &\leq G_j f(v_j). \end{aligned}$$

Using the triangle inequality for the discrepancy a second time, we finally obtain

$$ND_N((\mathbf{x}_{g(n)})_{n \geq 0}) \leq \sum_{j=0}^d a_j G_j f(v_j) \leq \sum_{j=0}^d \frac{N_{j+1}}{N_j} G_j f(v_j).$$

□

4 Indexing by the q -ary sum-of-digits function

We would now like to show results regarding index-transformed uniformly distributed sequences indexed by the q -ary sum-of-digits function. We first discuss an application of the general result in Theorem 1 (Section 4.1) to Halton- and (t, s) -sequences, and then show a refined result that applies to the particular case of van der Corput-sequences (Section 4.2).

4.1 Results for Halton- and (t, s) -sequences

Let $q \geq 2$ be an integer and $g(n) = s_q(n)$ the q -ary sum-of-digits function. For $j \in \mathbb{N}_0$ choose $N_j = q^j$. Then we have

$$G_{0,j}(k) = \#\{n : 0 \leq n < q^j, s_q(n) = k\}$$

and

$$(1 + x + x^2 + \dots + x^{q-1})^j = \sum_{k \in \mathbb{N}_0} G_{0,j}(k) x^k,$$

by expanding the polynomial on the left hand side of the latter equation. Hence the sequence $(G_{0,j}(k))_{k \in \mathbb{N}_0}$ is the j -fold convolution of the sequence $(\underbrace{1, 1, \dots, 1}_{q\text{-times}}, 0, 0, \dots)$, which

implies by [25, Theorem 1] that $G_{0,j}(k)$ is unimodal for sufficiently large j . Since any $n \in \mathbb{N}_0$ with $Aq^j \leq n < (A+1)q^j$ can be written as $n = n' + Aq^j$, where $0 \leq n' < q^j$, it follows that $s_q(n) = s_q(n') + s_q(A)$ and hence $G_{A,j}(k) = G_{0,j}(k - s_q(A))$, where we set $G_{0,j}(k - s_q(A)) := 0$ if $k < s_q(A)$. Consequently, $G_{A,j}(k)$ is unimodal for any $A \in \mathbb{N}_0$ and for sufficiently large j .

We recall the following lemma from [8].

Lemma 1 (Drmota and Larcher, [8, Lemma 1]). *For integers $q \geq 2$, $j \geq 1$, and $0 \leq k \leq j(q-1)$ we have*

$$G_{0,j}(k) = \frac{q^j}{\sqrt{2\pi j} \sigma_q} \exp\left(-\frac{x_{j,k}^2}{2}\right) \left(1 + \frac{P_1(x_{j,k})}{\sqrt{j}} + \frac{P_2(x_{j,k})}{j}\right) + O\left(\frac{q^j}{j^2}\right),$$

where $P_1(x)$ and $P_2(x)$ are polynomials, $P_1(x)$ is odd, where $x_{j,k} := \frac{k - \frac{j(q-1)}{2}}{\sigma_q \sqrt{j}}$, and where $\sigma_q := \sqrt{\frac{q^2-1}{12}}$. The implied constant in the O -notation is uniform for all k and only depends on q .

Due to Lemma 1, there exists some $c_q > 0$ such that for sufficiently large j we have $G_{A,j}(k) \leq c_q q^j / \sqrt{j}$, uniformly in k and A . Thus we obtain

$$G_j \leq c_q \frac{q^j}{\sqrt{j}} \tag{2}$$

for sufficiently large j . On the other hand, for $\tilde{k} = \lfloor j \frac{q-1}{2} \rfloor$ it follows that

$$\max_{k \in \mathbb{N}_0} G_{0,j}(k) \geq G_{0,j}(\tilde{k}) \geq c'_q \frac{q^j}{\sqrt{j}}. \tag{3}$$

Furthermore it is clear that $v_0 = 1$ and $v_j \leq qj$ for all $j \in \mathbb{N}$. As an application of Theorem 1, we obtain the following result.

Theorem 2. *Let $X := (\mathbf{x}_n)_{n \geq 0}$ be an s -dimensional sequence such that $m\tilde{D}_m((\mathbf{x}_n)_{n \geq 0}) \leq C(\log m)^s$ for all $m \in \mathbb{N}$, where C may depend on s or on the sequence X , but not on m . Let $q \geq 2$ be an integer. Then there exist $c_q^{(2)}, c_q^{(3)} > 0$, where $c_q^{(3)}$ may also depend on s and X , such that*

$$\frac{c_q^{(2)}}{\sqrt{\log N}} \leq D_N((\mathbf{x}_{s_q(n)})_{n \geq 0}) \leq c_q^{(3)} \frac{(\log \log N)^s}{\sqrt{\log N}}.$$

Proof. Assume that $q^d \leq N < q^{d+1}$. Then we obtain from Theorem 1 and Equation (3) that

$$D_N((\mathbf{x}_{s_q(n)})_{n \geq 0}) \geq \frac{c'_q}{N} \frac{q^d}{\sqrt{d}} \geq \frac{c_q^{(2)}}{\sqrt{\log N}}.$$

On the other hand, from Theorem 1 and Equation (2) ,

$$\begin{aligned} D_N((\mathbf{x}_{s_q(n)})_{n \geq 0}) &\leq \frac{1}{N} \sum_{j=1}^d q c_q \frac{q^j}{\sqrt{j}} C(\log(qj))^s \\ &\ll_q (\log d)^s \left(\frac{1}{N} \sum_{1 \leq j < d/2} \frac{q^j}{\sqrt{j}} + \frac{1}{N} \sum_{d/2 \leq j \leq d} \frac{q^j}{\sqrt{j}} \right) \\ &\ll_q (\log d)^s \left(\frac{\sqrt{\log N}}{\sqrt{N}} + \frac{1}{\sqrt{d}} \right) \\ &\ll_q \frac{(\log \log N)^s}{\sqrt{\log N}}, \end{aligned}$$

and the result follows. \square

The general lower bound in Theorem 2 is best possible with respect to the order of magnitude in N . This will follow from Theorem 3 below which deals with van der Corput-sequences.

There are several examples of sequences X which satisfy the conditions in Theorem 2 such as Halton- or (t, s) -sequences (for a proof of this fact, we refer to Section 6 of this paper). We thus obtain the following corollary.

Corollary 1. *Let $q \geq 2$ be an integer.*

1. *Let $(\mathbf{x}_n)_{n \geq 0}$ be an s -dimensional Halton-sequence in pairwise co-prime bases b_1, \dots, b_s . Then there exist $c_q^{(2)}, c_{q,s,b_1,\dots,b_s}^{(4)} > 0$ such that*

$$\frac{c_q^{(2)}}{\sqrt{\log N}} \leq D_N((\mathbf{x}_{s_q(n)})_{n=0}^{N-1}) \leq c_{q,s,b_1,\dots,b_s}^{(4)} \frac{(\log \log N)^s}{\sqrt{\log N}}.$$

2. *Let $(\mathbf{x}_n)_{n \geq 0}$ be a (t, s) -sequence in base b . Then there exist $c_q^{(2)}, c_{q,b,s,t}^{(5)} > 0$ such that*

$$\frac{c_q^{(2)}}{\sqrt{\log N}} \leq D_N((\mathbf{x}_{s_q(n)})_{n \geq 0}) \leq c_{q,b,s,t}^{(5)} \frac{(\log \log N)^s}{\sqrt{\log N}}.$$

The result of the first part of Corollary 1 can be improved for the special instance of van der Corput-sequences, as we will show next.

4.2 The van der Corput-sequence indexed by the sum-of-digits function

The following results are based on a general discrepancy estimate which was first presented by Hellekalek [14]. The following definitions stem from [14, 15, 17]. We refer to these references for further information.

For an integer $b \geq 2$ let $\mathbb{Z}_b = \{z = \sum_{r=0}^{\infty} z_r b^r : z_r \in \{0, \dots, b-1\}\}$ be the set of b -adic numbers. \mathbb{Z}_b forms an abelian group under addition. The set \mathbb{N}_0 is a subset of \mathbb{Z}_b . The *Monna map* $\phi_b : \mathbb{Z}_b \rightarrow [0, 1)$ is defined by

$$\phi_b(z) := \sum_{r=0}^{\infty} \frac{z_r}{b^{r+1}}.$$

Note that the radical inverse function φ_b is nothing but ϕ_b restricted to \mathbb{N}_0 . We also define the inverse $\phi_b^+ : [0, 1) \rightarrow \mathbb{Z}_b$ by

$$\phi_b^+ \left(\sum_{r=0}^{\infty} \frac{x_r}{b^{r+1}} \right) := \sum_{r=0}^{\infty} x_r b^r,$$

where we always use the finite b -adic representation for b -adic rationals in $[0, 1)$.

For $k \in \mathbb{N}_0$ we can define characters $\chi_k : \mathbb{Z}_b \rightarrow \{c \in \mathbb{C} : |c| = 1\}$ of \mathbb{Z}_b by

$$\chi_k(z) = \exp(2\pi i \phi_b(k)z).$$

Finally, let $\gamma_k : [0, 1) \rightarrow \{c \in \mathbb{C} : |c| = 1\}$ where $\gamma_k(x) = \chi_k(\phi_b^+(x))$.

For $b \geq 2$ we put $\rho_b(0) = 1$ and $\rho_b(k) = \frac{2}{b^{r+1} \sin(\pi \kappa_r / b)}$ for $k \in \mathbb{N}$ with base b expansion $k = \kappa_0 + \kappa_1 b + \dots + \kappa_r b^r$, $\kappa_r \neq 0$.

We have the following general discrepancy bound which is based on the functions γ_k .

Lemma 2. *Let $g \in \mathbb{N}$. For any sequence $(y_n)_{n \geq 0}$ in $[0, 1)$ we have*

$$D_N((y_n)_{n \geq 0}) \leq \frac{1}{b^g} + \sum_{k=1}^{b^g-1} \rho_b(k) \left| \frac{1}{N} \sum_{n=0}^{N-1} \gamma_k(y_n) \right|.$$

Proof. For the special case of a prime b , this result was shown by Hellekalek [14, Theorem 3.6]. Using [17, Lemma 2.10 and 2.11] it is easy to see that Hellekalek's result can be generalized to the one given in the lemma (cf. [16]). \square

We show a discrepancy bound for the van der Corput-sequence indexed by the q -ary sum-of-digits function for small values of q . This result improves on the first part of Corollary 1 for van der Corput-sequences. Moreover, it shows that the general lower bound from Theorem 2 is best possible in the order of magnitude in N .

Theorem 3. *Let $b, q \geq 2$ be integers with $q < 14$, let $(x_n)_{n \geq 0}$ be the van der Corput-sequence in base b and let $(s_q(n))_{n \geq 0}$ be the sequence of the q -adic sum-of-digits function. Then we have*

$$D_N((x_{s_q(n)})_{n \geq 0}) \ll_{b,q} \frac{1}{\sqrt{\log N}}.$$

Remark 1. In view of Theorem 2, the upper bound in Theorem 3 is best possible with respect to the order of magnitude in N .

Before we give the proof of Theorem 3, we need some preparations and auxiliary results. Writing $e(x) := \exp(2\pi i x)$ for short, we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \gamma_k(x_{s_q(n)}) = \frac{1}{N} \sum_{n=0}^{N-1} e(s_q(n) \phi_b(k)) =: T_k(N).$$

Lemma 3. Let $b, q \geq 2$ be integers, let $k \in \mathbb{N}$ and let $(x_n)_{n \geq 0}$ be the van der Corput-sequence in base b . Then for any $m \in \mathbb{N}_0$ it is true that

$$|T_k(q^m)| \leq \left(1 - \frac{16(q-1)}{q^2} \|\phi_b(k)\|^2\right)^{m/2},$$

where $\|x\|$ is the distance of a real x to the nearest integer.

Proof. First observe that

$$T_k(q^m) = \frac{1}{q^m} \sum_{n_0, \dots, n_{m-1}=0}^{q-1} e((n_0 + \dots + n_{m-1})\phi_b(k)) = (T_k(q))^m.$$

We now proceed as in [27]. We use the identities $\exp(ix) + \exp(-ix) = 2\cos x$ and $\cos(2x) = 1 - 2\sin^2 x$ to obtain

$$\begin{aligned} |T_k(q)|^2 &= \frac{1}{q^2} \sum_{n, n'=0}^{q-1} e((n - n')\phi_b(k)) \\ &= \frac{1}{q^2} \left(q + \sum_{\substack{n, n'=0 \\ n < n'}}^{q-1} (e((n - n')\phi_b(k)) + e(-(n - n')\phi_b(k))) \right) \\ &= \frac{1}{q^2} \left(q + 2 \sum_{\substack{n, n'=0 \\ n < n'}}^{q-1} \cos(2\pi(n - n')\phi_b(k)) \right) \\ &= \frac{1}{q^2} \left(q + 2 \sum_{\substack{n, n'=0 \\ n < n'}}^{q-1} (1 - 2\sin^2(\pi(n - n')\phi_b(k))) \right) \\ &= 1 - \frac{4}{q^2} \sum_{\substack{n, n'=0 \\ n < n'}}^{q-1} \sin^2(\pi(n - n')\phi_b(k)) \\ &\leq 1 - \frac{4(q-1)}{q^2} \sin^2(\pi\phi_b(k)) \\ &\leq 1 - \frac{16(q-1)}{q^2} \|\phi_b(k)\|^2, \end{aligned}$$

Therefore,

$$|T_k(q^m)| \leq \left(1 - \frac{16(q-1)}{q^2} \|\phi_b(k)\|^2\right)^{m/2}.$$

□

We also need the following lemma.

Lemma 4. For $k \in \mathbb{N}$ and any $N \in \mathbb{N}$ with q -adic expansion $N = \sum_{r=0}^R a_r q^r$ we have

$$|T_k(N)| \leq \frac{1}{N} \sum_{r=0}^R a_r q^r |T_k(q^r)|.$$

Proof. For $N = \sum_{r=0}^R a_r q^r$,

$$\{0, \dots, N-1\} = \bigcup_{r=0}^R \{a_R q^R + \dots + a_{r+1} q^{r+1}, \dots, a_R q^R + \dots + a_r q^r - 1\},$$

and hence

$$\begin{aligned} N|T_k(N)| &= \left| \sum_{n=0}^{N-1} e(s_q(n)\phi_b(k)) \right| \\ &= \left| \sum_{r=0}^R e((a_R + \dots + a_{r+1})\phi_b(k)) \sum_{n=0}^{a_r q^r - 1} e(s_q(n)\phi_b(k)) \right| \\ &\leq \sum_{r=0}^R \left| \sum_{n=0}^{a_r q^r - 1} e(s_q(n)\phi_b(k)) \right| \\ &= \sum_{r=0}^R \left| \sum_{u=0}^{a_r - 1} e(u\phi_b(k)) \sum_{n=0}^{q^r - 1} e(s_q(n)\phi_b(k)) \right| \\ &\leq \sum_{r=0}^R a_r \left| \sum_{n=0}^{q^r - 1} e(s_q(n)\phi_b(k)) \right| \\ &= \sum_{r=0}^R a_r q^r |T_k(q^r)|. \end{aligned}$$

□

We are now ready to give the proof of Theorem 3.

Proof. For $k \in \{b^r, \dots, b^{r+1} - 1\}$ we have $\varphi_b(k) = \frac{A_k}{b^{r+1}}$ with $A_k \in \{1, \dots, b^{r+1} - 1\}$, where $A_{k_1} \neq A_{k_2}$ for $k_1 \neq k_2$. Hence we obtain from Lemma 3

$$\begin{aligned} \sum_{k=1}^{b^g-1} \rho_b(k) |T_k(q^m)| &\leq \sum_{r=0}^{g-1} \frac{2}{b^{r+1} \sin(\pi/b)} \sum_{k=b^r}^{b^{r+1}-1} \left(1 - \frac{16(q-1)}{q^2} \left\| \frac{A_k}{b^{r+1}} \right\|^2 \right)^{m/2} \\ &\leq \sum_{r=0}^{g-1} \frac{2}{b^{r+1} \sin(\pi/b)} \sum_{a=1}^{b^{r+1}-1} \left(1 - \frac{16(q-1)}{q^2} \left\| \frac{a}{b^{r+1}} \right\|^2 \right)^{m/2}. \end{aligned}$$

For the inner sum we have

$$\begin{aligned} &\sum_{a=1}^{b^{r+1}-1} \left(1 - \frac{16(q-1)}{q^2} \left\| \frac{a}{b^{r+1}} \right\|^2 \right)^{m/2} \\ &= \sum_{1 \leq a < b^{r+1}/2} \left(1 - \frac{16(q-1)}{q^2} \frac{a^2}{b^{2r+2}} \right)^{m/2} \\ &\quad + \sum_{b^{r+1}/2 \leq a < b^{r+1}} \left(1 - \frac{16(q-1)}{q^2} \left(1 - \frac{a}{b^{r+1}} \right)^2 \right)^{m/2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{b^{m(r+1)}} \sum_{1 \leq a < b^{r+1}/2} \left(b^{2r+2} - \frac{16(q-1)}{q^2} a^2 \right)^{m/2} \\
&\quad + \frac{1}{b^{m(r+1)}} \sum_{b^{r+1}/2 \leq a < b^{r+1}} \left(b^{2r+2} - \frac{16(q-1)}{q^2} (b^{r+1} - a)^2 \right)^{m/2} \\
&= \frac{2}{b^{m(r+1)}} \sum_{1 \leq a < b^{r+1}/2} \left(b^{2r+2} - \frac{16(q-1)}{q^2} a^2 \right)^{m/2} + \delta(b) \left(1 - \frac{4(q-1)}{q^2} \right)^{m/2},
\end{aligned}$$

where $\delta(b) = 0$ when b is odd and $\delta(b) = 1$ when b is even.

The assumption $q < 14$ yields $\frac{16(q-1)}{q^2} \geq 1$, and hence

$$\begin{aligned}
\sum_{a=1}^{b^{r+1}-1} \left(1 - \frac{16(q-1)}{q^2} \left\| \frac{a}{b^{r+1}} \right\|^2 \right)^{m/2} &\leq \frac{2}{b^{m(r+1)}} \sum_{1 \leq a < b^{r+1}/2} (b^{2r+2} - a^2)^{m/2} + \left(\frac{3}{4} \right)^{m/2} \\
&\leq \frac{2}{b^{m(r+1)}} \sum_{u=1}^{b^{2r+2}-1} u^{m/2} + \left(\frac{3}{4} \right)^{m/2} \\
&\leq \frac{2}{b^{m(r+1)}} \int_1^{b^{2r+2}} u^{m/2} du + \left(\frac{3}{4} \right)^{m/2} \\
&\ll_{b,q} \frac{b^{2r+2}}{m+1} + \left(\frac{3}{4} \right)^{m/2}
\end{aligned}$$

with an implied constant depending only on b and q . Therefore

$$\sum_{k=1}^{b^g-1} \rho_b(k) |T_k(q^m)| \ll_{b,q} \sum_{r=0}^{g-1} \frac{1}{b^{r+1}} \left(\frac{b^{2(r+1)}}{m+1} + \left(\frac{3}{4} \right)^{m/2} \right) \ll_{b,q} \frac{b^g}{m+1}, \quad (4)$$

again with implied constants depending only on b and q .

Assume that $N = \sum_{r=0}^R a_r q^r$. Then, using Lemma 4 and (4), we obtain

$$\begin{aligned}
\sum_{k=1}^{b^g-1} \rho_b(k) |T_k(N)| &\leq \frac{1}{N} \sum_{m=0}^R a_m q^m \sum_{k=1}^{b^g-1} \rho_b(k) |T_k(q^m)| \\
&\ll_{b,q} b^g \frac{1}{N} \sum_{m=0}^R a_m \frac{q^m}{m+1}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{N} \sum_{m=0}^R a_m \frac{q^m}{m+1} &\leq \frac{1}{N} \sum_{m=0}^{\lfloor R/2 \rfloor} a_m q^m + \frac{1}{N} \sum_{m=\lfloor R/2 \rfloor+1}^R a_m \frac{q^m}{m+1} \\
&\ll_q \frac{q^{R/2}}{N} + \frac{1}{R} \ll_q \frac{1}{\log N}
\end{aligned}$$

we obtain

$$\sum_{k=1}^{b^g-1} \rho_b(k) |T_k(N)| \ll_{b,q} \frac{b^g}{\log N}.$$

From Lemma 2 it follows that

$$D_N((x_{s_q(n)})_{n \geq 0}) \ll_{b,q} \frac{1}{b^g} + \frac{b^g}{\log N}.$$

Choosing $g = \lfloor \log_b \sqrt{\log N} \rfloor$ yields

$$D_N((x_{s_q(n)})_{n \geq 0}) \ll_{b,q} \frac{1}{\sqrt{\log N}}.$$

□

Remark 2. We remark that, in principle, the method of proof based on Lemma 2 can not only be used for van der Corput-sequences, but also for Halton-sequences in higher dimensions. However, this leads to a discrepancy bound of order $(\log N)^{-\frac{1}{s+1}}$, which is considerably weaker than the one presented in Theorem 2.

5 Other index-transformations

In this section, we would now like to discuss index-transformed Halton- and digital (t, s) -sequences indexed by a different kind of sequence than the sum-of-digits function, as, e.g., $(\lfloor n^\alpha \rfloor)_{n \geq 0}$ with $0 < \alpha < 1$. The following theorem provides another general result, namely lower and upper bounds on the discrepancy of sequences indexed by functions which in some sense are “moderately” monotonically increasing.

Theorem 4. *Let $A \in \mathbb{N}_0$ and write $\mathbb{N}_A := \{A, A+1, A+2, \dots\}$. Let $f : \mathbb{N}_0 \rightarrow \mathbb{N}_A$ be surjective and monotonically increasing. Moreover, define, for $k \in \mathbb{N}_A$,*

$$F(k) := \#\{n : n \in \mathbb{N}_0, f(n) = k\}.$$

Under the assumption that $F(k)$ is monotonically increasing in k for sufficiently large k , the following three assertions hold.

1. *For an arbitrary sequence $(\mathbf{x}_n)_{n \geq 0}$ in $[0, 1]^s$ it is true that*

$$\frac{F(f(N) - 1)}{N} \leq D_N((\mathbf{x}_{f(n)})_{n \geq 0}).$$

2. *For a Halton-sequence $(\mathbf{x}_n)_{n \geq 0}$ in co-prime bases b_1, \dots, b_s ,*

$$D_N((\mathbf{x}_{f(n)})_{n \geq 0}) \leq C \frac{2F(f(N-1) + 1)(\log N)^s}{N},$$

where C is a constant independent of N .

3. *For a digital (t, s) -sequence $(\mathbf{x}_n)_{n \geq 0}$ over \mathbb{F}_p for prime p ,*

$$D_N((\mathbf{x}_{f(n)})_{n \geq 0}) \leq \tilde{C} p^t \frac{2F(f(N-1) + 1)(\log N)^s}{N},$$

where \tilde{C} is a constant independent of N .

Proof. 1. Let $(\mathbf{x}_n)_{n \geq 0}$ be an arbitrary sequence in $[0, 1]^s$, and let f and F be as in the theorem. If $f(N) = A$, then, due to the properties of f , we obtain $F(f(N) - 1) = 0$, so the lower bound on the discrepancy is trivially fulfilled.

If, on the other hand, $f(N) > A$, then it follows by the surjectivity of f that there exist $n \in \mathbb{N}_0$ such that $f(n) = f(N) - 1$. Furthermore, whenever n is such that $f(n) = f(N) - 1 < f(N)$, it follows by the monotonicity of f that $n < N$. Hence, the value $f(N) - 1$ occurs $F(f(N) - 1)$ times among $f(0), \dots, f(N - 1)$, and the point $\mathbf{x}_{f(N)-1}$ is attained $F(f(N) - 1)$ times in the sequence $\mathbf{x}_{f(0)}, \dots, \mathbf{x}_{f(N-1)}$. The lower bound follows by considering an arbitrarily small interval containing $\mathbf{x}_{f(N)-1}$.

2. Without loss of generality, assume $f(0) = 0$, i.e., $A = 0$.

Furthermore, it is no loss of generality to assume that $f(1) = 1$ and that $F(k)$ is monotonically increasing in k for $k \geq 0$. Indeed, if this is not the case, we can disregard a suitable number of initial elements $\mathbf{x}_{f(0)}, \dots, \mathbf{x}_{f(N_0)}$, without changing the discrepancy of the first N points of the sequence $(\mathbf{x}_{f(n)})_{n \geq 0}$ by more than $\frac{N_0}{N}$.

Let $b_1, \dots, b_s \geq 2$ be co-prime integers and let $(\mathbf{x}_n)_{n \geq 0}$ be the corresponding Halton-sequence. For estimating the discrepancy, we consider an arbitrary interval

$$I := \prod_{i=1}^s [0, \alpha^{(i)}) \subseteq [0, 1]^s,$$

for some $\alpha^{(1)}, \dots, \alpha^{(s)} \in (0, 1]$. For each $i \in \{1, \dots, s\}$, choose m_i as the minimal integer such that $N \leq b_i^{m_i}$. Since $f(N - 1) \leq N - 1$, the i -th component $x_{f(n)}^{(i)}$ of a point $\mathbf{x}_{f(n)}$, $1 \leq i \leq s$, $0 \leq n \leq N - 1$, has at most m_i non-zero digits in its base b_i representation. From this, it is easily derived that we can restrict ourselves to considering only $\alpha^{(i)}$ with at most m_i non-zero digits in their base b_i expansion, $1 \leq i \leq s$, as this assumption changes $D_N((\mathbf{x}_{f(n)})_{n \geq 0})$ by a term of order of at most N^{-1} . We can therefore write I as the disjoint union of intervals

$$I(j_1, \dots, j_s) := \prod_{i=1}^s \left[\sum_{r=1}^{j_i-1} \frac{\alpha_r^{(i)}}{b_i^r}, \sum_{r=1}^{j_i} \frac{\alpha_r^{(i)}}{b_i^r} \right),$$

where $1 \leq j_i \leq m_i$ for $1 \leq i \leq s$ and the $\alpha_r^{(i)}$ represent the base b_i digits of $\alpha^{(i)}$. Each of the $I(j_1, \dots, j_s)$ can in turn be written as the disjoint union of intervals

$$\prod_{i=1}^s J(j_i, k_i) := \prod_{i=1}^s \left[\sum_{r=1}^{j_i-1} \frac{\alpha_r^{(i)}}{b_i^r} + \frac{k_i}{b_i^{j_i}}, \sum_{r=1}^{j_i-1} \frac{\alpha_r^{(i)}}{b_i^r} + \frac{k_i + 1}{b_i^{j_i}} \right),$$

with $1 \leq j_i \leq m_i$ and $0 \leq k_i \leq \alpha_{j_i}^{(i)} - 1$. If $\alpha_{j_i}^{(i)} = 0$, then $J(j_i, k_i)$ is of zero volume containing no points. Hence we can restrict ourselves to considering only those $J(j_i, k_i)$ with $\alpha_{j_i}^{(i)} \geq 1$.

Let now $i \in \{1, \dots, s\}$ and $v \geq 0$ be fixed. By the construction principle of the points of the Halton-sequence, we see that $x_v^{(i)}$ is contained in $J(j_i, k_i)$ if and only if

$$\begin{pmatrix} v_0^{(i)} \\ \vdots \\ v_{j_i-2}^{(i)} \\ v_{j_i-1}^{(i)} \end{pmatrix} = \begin{pmatrix} \alpha_i^{(1)} \\ \vdots \\ \alpha_i^{(j_i-1)} \\ k_i \end{pmatrix}, \quad (5)$$

where the $v_r^{(i)}$, $0 \leq r \leq j_i - 1$ are the digits of v in base b_i . Note that (5) has exactly one solution $(v_0^{(i)}, \dots, v_{j_i-1}^{(i)})$ modulo b_i . Hence we can identify exactly one remainder $R^{(i)}$ modulo $b_i^{j_i}$, such that $x_v^{(i)} \in J(j_i, k_i)$ if and only if $v \equiv R^{(i)} \pmod{b_i^{j_i}}$. By the Chinese Remainder Theorem, there exists exactly one remainder R modulo $Q := \prod_{i=1}^s b_i^{j_i}$ such that

$$\mathbf{x}_v \in \prod_{i=1}^s J(j_i, k_i) \text{ if and only if } v \equiv R \pmod{Q}.$$

We now deduce an estimate for the number of points among $\mathbf{x}_{f(0)}, \dots, \mathbf{x}_{f(N-1)}$ that are contained in an interval of the type $\prod_{i=1}^s J(j_i, k_i)$. For short, we denote this number by $A(\prod_{i=1}^s J(j_i, k_i))$.

Note that there exists a number $\theta = \theta(R, Q, f(N-1)) \in \{0, 1\}$ such that $0 = f(0) \leq R + wQ \leq f(N-1)$ if and only if $w \in \{0, \dots, \lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta\}$, so

$$A\left(\prod_{i=1}^s J(j_i, k_i)\right) \geq \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(R + wQ) \geq \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ), \quad (6)$$

where we used the monotonicity of F . On the other hand, with the same argument,

$$A\left(\prod_{i=1}^s J(j_i, k_i)\right) \leq \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta} F(R + wQ) \leq \sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor + \theta} F(wQ). \quad (7)$$

For the following, let $K = \lfloor \frac{f(N-1)}{Q} \rfloor + \theta$. Let

$$\Sigma_A := \sum_{r=0}^{(K-1)Q-1} F(r),$$

and note that we can write

$$\Sigma_A = \sum_{w=0}^{K-2} \sum_{r=0}^{Q-1} F(wQ + r) \geq Q \sum_{w=0}^{K-2} F(wQ) = Q \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ).$$

On the other hand, by the definition of θ ,

$$\Sigma_A = \sum_{r=0}^{(\lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta)Q - 1} F(r) \leq \sum_{r=0}^{f(N-1)-1} F(r) \leq N - 1,$$

from which we conclude that

$$\sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ) \leq \frac{N-1}{Q}. \quad (8)$$

Moreover, let

$$\Sigma_B := \sum_{r=1}^{KQ} F(r),$$

for which we can derive, in the same way as the corresponding estimate for Σ_A ,

$$\Sigma_B \leq Q \sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor + \theta} F(wQ).$$

Again by the definition of θ ,

$$\Sigma_B = \sum_{r=1}^{(\lfloor \frac{f(N-1)}{Q} \rfloor + \theta)Q} F(r) \geq \sum_{r=1}^{f(N-1)} F(r) = \#\{n \in \mathbb{N}_0 : 0 < f(n) \leq f(N-1)\} \geq N-1,$$

where we used that $f(1) = 1$ and that f is monotonically increasing. Consequently,

$$\sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor + \theta} F(wQ) \geq \frac{N-1}{Q}. \quad (9)$$

Note, furthermore, that

$$\begin{aligned} 0 \leq \sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor + \theta} F(wQ) - \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ) &\leq F\left(\left(\left\lfloor \frac{f(N-1)}{Q} \right\rfloor - 1 + \theta\right)Q\right) \\ &\quad + F\left(\left(\left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta\right)Q\right) \\ &\leq 2F(f(N-1) + 1). \end{aligned} \quad (10)$$

Combining Equations (6), (9), and (10), and noting that $\lambda(\prod_{i=1}^s J(j_i, k_i)) = \frac{1}{Q}$, gives

$$\begin{aligned} \frac{1}{N} A\left(\prod_{i=1}^s J(j_i, k_i)\right) - \frac{1}{Q} &\geq \frac{1}{N} \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ) - \frac{1}{Q} \\ &\geq \frac{\sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor + \theta} F(wQ) - 2F(f(N-1) + 1)}{N} - \frac{1}{Q} \\ &\geq \frac{-2F(f(N-1) + 1)}{N} + \frac{N-1}{QN} - \frac{1}{Q} \\ &\geq \frac{-2F(f(N-1) + 1)}{N} - \frac{1}{NQ}. \end{aligned}$$

In exactly the same way, using (7), (8), and (10), we get

$$\frac{1}{N} A\left(\prod_{i=1}^s J(j_i, k_i)\right) - \frac{1}{Q} \leq \frac{2F(f(N-1) + 1)}{N} + \frac{1}{NQ},$$

from which we derive

$$\left| \frac{1}{N} A \left(\prod_{i=1}^s J(j_i, k_i) \right) - \frac{1}{Q} \right| \leq \frac{2F(f(N-1)+1)}{N} + \frac{1}{NQ}.$$

Finally, note that, by writing $A(I)$ for the number of points of $(\mathbf{x}_{f(n)})_{n=0}^{N-1}$ in I ,

$$\begin{aligned} \left| \frac{A(I)}{N} - \lambda(I) \right| &\leq \\ &\leq \sum_{j_1=1}^{m_1} \cdots \sum_{j_s=1}^{m_s} \sum_{k_1=0}^{\alpha_{j_1}^{(1)}-1} \cdots \sum_{k_s=0}^{\alpha_{j_s}^{(s)}-1} \left| \frac{1}{N} A \left(\prod_{i=1}^s J(j_i, k_i) \right) - \lambda \left(\prod_{i=1}^s J(j_i, k_i) \right) \right| \\ &\leq C \frac{(\log N)^s F(f(N-1)+1)}{N}, \end{aligned}$$

for a suitably chosen constant C , and the result follows.

3. As in Item 2, assume without loss of generality that $f(0) = 0$, $f(1) = 1$, and that $F(k)$ is monotonically increasing in k for $k \geq 1$.

Let p be a prime and let $(\mathbf{x}_n)_{n \geq 0}$ be a digital (t, s) -sequence over \mathbb{F}_p . For estimating the discrepancy, we consider an arbitrary interval

$$I := \prod_{i=1}^s [0, \alpha^{(i)}) \subseteq [0, 1)^s,$$

for some $\alpha^{(1)}, \dots, \alpha^{(s)} \in (0, 1]$. Choose m as the minimal integer such that $N \leq p^m$. By a similar argument as for the case of Halton sequences, we can restrict ourselves to considering only $\alpha^{(i)}$ with at most m non-zero digits $\alpha_1^{(i)}, \dots, \alpha_m^{(i)}$ in their base p expansion. Moreover, with the same reasoning as in the Halton case, we see that we essentially only need to deal with intervals of the form

$$\prod_{i=1}^s J(j_i, k_i) := \prod_{i=1}^s \left[\sum_{r=1}^{j_i-1} \frac{\alpha_r^{(i)}}{p^r} + \frac{k_i}{p^{j_i}}, \sum_{r=1}^{j_i-1} \frac{\alpha_r^{(i)}}{p^r} + \frac{k_i+1}{p^{j_i}} \right),$$

with $1 \leq j_i \leq m$ and $0 \leq k_i \leq \alpha_{j_i}^{(i)} - 1$. Again, if $\alpha_{j_i}^{(i)} = 0$, then $J(j_i, k_i)$ is of zero volume containing no points, so we can restrict ourselves to considering only those $J(j_i, k_i)$ with $\alpha_{j_i}^{(i)} \geq 1$.

As for the case of Halton sequences, we would like to derive an upper and a lower bound on the number $A(\prod_{i=1}^s J(j_i, k_i))$ of points contained in $\prod_{i=1}^s J(j_i, k_i)$. To this end, denote the r -th row of a generator matrix C_j , $1 \leq j \leq s$ of $(\mathbf{x}_n)_{n \geq 0}$ by $\mathbf{c}_r^{(j)}$. For an integer $v \geq 0$, the point \mathbf{x}_v is contained in $\prod_{i=1}^s J(j_i, k_i)$ if and only if

$$\mathcal{C} \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{pmatrix} = A^\top, \tag{11}$$

where v_0, v_1, v_2, \dots are the base p digits of v , where

$$A := (\alpha_1^{(1)}, \dots, \alpha_{j_1-1}^{(1)}, k_1, \alpha_1^{(2)}, \dots, \alpha_{j_2-1}^{(2)}, k_2, \dots, \alpha_1^{(s)}, \dots, \alpha_{j_s-1}^{(s)}, k_s) \in \mathbb{F}_p^{j_1+\dots+j_s},$$

and

$$\mathcal{C} := (\mathbf{c}_1^{(1)}, \dots, \mathbf{c}_{j_1}^{(1)}, \mathbf{c}_1^{(2)}, \dots, \mathbf{c}_{j_2}^{(2)}, \dots, \mathbf{c}_1^{(s)}, \dots, \mathbf{c}_{j_s}^{(s)})^\top \in \mathbb{F}_p^{(j_1+\dots+j_s) \times \mathbb{N}}.$$

Let now $Q := p^{j_1+\dots+j_s+t}$, let $w \in \mathbb{N}_0$ and consider those $v \geq 0$ with $wQ \leq v \leq (w+1)Q-1$. For these v , the first $j_1+j_2+\dots+j_s+t$ digits in their base p expansion vary, while all the other digits are fixed. Hence we can write (11) as

$$D_1 \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{j_1+\dots+j_s+t} \end{pmatrix} + D_2 \cdot \begin{pmatrix} v_{j_1+\dots+j_s+t+1} \\ v_{j_1+\dots+j_s+t+2} \\ \vdots \end{pmatrix} = A^\top,$$

where $\mathcal{C} = (D_1 | D_2)$ and where D_1 is an $(j_1 + \dots + j_s) \times (j_1 + \dots + j_s + t)$ -matrix and D_2 is an $(j_1 + \dots + j_s) \times \mathbb{N}$ -matrix over \mathbb{F}_p .

Due to the fact that $(\mathbf{x}_n)_{n \geq 0}$ is a digital (t, s) -sequence, it follows that D_1 has full rank, and hence there are exactly p^t values v in $\{wQ, wQ+1, \dots, (w+1)Q-1\}$ such that \mathbf{x}_v is contained in $\prod_{i=1}^s J(j_i, k_i)$.

Now note again that there exists a number $\theta = \theta(Q, f(N-1)) \in \{0, 1\}$ such that $0 = f(0) \leq wQ \leq f(N-1)$ if and only if $w \in \{0, \dots, \lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta\}$. By our observations above, for each of these $w \in \{0, \dots, \lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta\}$ there exist p^t integers $R_{w,1}, \dots, R_{w,p^t} \in \{0, \dots, Q-1\}$ such that exactly the points $\mathbf{x}_{R_{w,1}+wQ}, \dots, \mathbf{x}_{R_{w,p^t}+wQ}$ among $\mathbf{x}_{wQ}, \mathbf{x}_{wQ+1}, \dots, \mathbf{x}_{(w+1)Q-1}$ are contained in $\prod_{i=1}^s J(j_i, k_i)$. Therefore, we can estimate

$$A \left(\prod_{i=1}^s J(j_i, k_i) \right) \geq \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} \sum_{z=1}^{p^t} F(R_{w,z} + wQ) \geq p^t \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ), \quad (12)$$

and

$$A \left(\prod_{i=1}^s J(j_i, k_i) \right) \leq \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta} \sum_{z=1}^{p^t} F(R_{w,z} + wQ) \leq p^t \sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor + \theta} F(wQ). \quad (13)$$

In exactly the same way as for a Halton sequence, we obtain, by noting that $\lambda(\prod_{i=1}^s J(j_i, k_i)) = \frac{1}{p^{1+\dots+1+s}} = \frac{p^t}{Q}$,

$$\left| \frac{1}{N} A \left(\prod_{i=1}^s J(j_i, k_i) \right) - \frac{1}{Q} \right| \leq \frac{p^t 2F(f(N-1)+1)}{N} + \frac{p^t}{NQ},$$

and the result follows. \square

Examples of functions f and F satisfying the assumptions of Theorem 4 are obtained as follows. Let $g : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ be a function that is twice differentiable on $(0, \infty)$, with $g'(x) > 0$ and $g''(x) < 0$ for $x \in (0, \infty)$. Moreover, define $f(n) := \lfloor g(n) \rfloor$ for $n \in \mathbb{N}$. It then easily follows that f and F indeed fulfill the assumptions of the theorem and we obtain

$$F(k+1) = \lceil g^{-1}(k+1) \rceil - \lceil g^{-1}(k) \rceil. \quad (14)$$

We thus obtain the following exemplary corollary to Theorem 4.

Corollary 2. *Let $\alpha \in (0, 1)$. Then the following assertions hold.*

1. *For a Halton-sequence $(\mathbf{x}_n)_{n \geq 0}$ in co-prime bases b_1, \dots, b_s ,*

$$\overline{C}_1 \frac{1}{N^\alpha} \leq D_N((\mathbf{x}_{\lfloor n^\alpha \rfloor})_{n \geq 0}) \leq \overline{C}_2 \frac{(\log N)^s}{N^\alpha},$$

where $\overline{C}_1, \overline{C}_2$ are constants that depend on the sequence and on α , but are independent of N .

2. *For a digital (t, s) -sequence $(\mathbf{x}_n)_{n \geq 0}$ over \mathbb{Z}_p for prime p ,*

$$\overline{\overline{C}}_1 \frac{1}{N^\alpha} \leq D_N((\mathbf{x}_{\lfloor n^\alpha \rfloor})_{n \geq 0}) \leq \overline{\overline{C}}_2 \frac{(\log N)^s}{N^\alpha},$$

where $\overline{\overline{C}}_1, \overline{\overline{C}}_2$ are constants that depend on the sequence and on α , but are independent of N .

Proof. The result follows by combining Theorem 2 with the observation that

$$c'_\alpha k^{\frac{1}{\alpha}-1} \leq F(k) \leq c_\alpha k^{\frac{1}{\alpha}-1},$$

with constants $c'_\alpha, c_\alpha > 0$ that depend on α , but not on k . □

6 Appendix: Uniform discrepancy

In Corollary 1 we implicitly used the fact that (t, s) -sequences in base b as well as Halton-sequences in pairwise co-prime bases b_1, \dots, b_s have uniform discrepancy of order $(\log N)^s/N$. Since we are not aware of a proof of these facts in the existing literature, we provide one here.

6.1 Uniform discrepancy of (t, s) -sequences in base b

Assume that $\Delta_b(t, m, s)$ is a number for which

$$b^m D_{b^m}(\mathcal{P}) \leq \Delta_b(t, m, s)$$

holds for the discrepancy of any (t, m, s) -net \mathcal{P} in base b .

Theorem 5. *Let $(\mathbf{x}_n)_{n \geq 0}$ be a (t, s) -sequence in base b . Then we have*

$$N \widetilde{D}_N((\mathbf{x}_n)_{n \geq 0}) \leq (2b-1) \left(tb^t + \sum_{m=t}^{\lfloor \log_b N \rfloor} \Delta_b(t, m, s) \right).$$

Proof. Let $k \in \mathbb{N}_0$. We show that

$$ND_N((\mathbf{x}_{n+k})_{n \geq 0}) \leq (2b-1) \left(tb^t + \sum_{m=t}^{\lfloor \log_b N \rfloor} \Delta_b(t, m, s) \right)$$

uniformly in $k \in \mathbb{N}_0$.

For $N < b^t$, the assertion follows trivially by $ND_N((\mathbf{x}_{n+k})_{n \geq 0}) \leq N$.

Let now $N \in \mathbb{N}$, $N \geq b^t$ with b -adic expansion $N = a_r b^r + a_{r-1} b^{r-1} + \dots + a_1 b + a_0$ where $a_j \in \{0, \dots, b-1\}$ for $0 \leq j \leq r$ and $a_r \neq 0$ (note that $r \geq t$). For given $k \in \mathbb{N}_0$, choose $\ell \in \mathbb{N}$ such that $(\ell-1)b^r \leq k < \ell b^r$. Then we can write

$$k = \ell b^r - (d_{r-1} b^{r-1} + \dots + d_1 b + d_0) - 1$$

with some $d_j \in \{0, \dots, b-1\}$ for $0 \leq j \leq r-1$, and

$$k = (\ell-1)b^r + \kappa_{r-1} b^{r-1} + \dots + \kappa_1 b + \kappa_0$$

with some $\kappa_j \in \{0, \dots, b-1\}$ for $0 \leq j \leq r-1$. Note that therefore $d_j + \kappa_j = (b-1)$ for $0 \leq j < r$.

We split up the point set $\mathcal{P}_{k,N} := \{\mathbf{x}_n : k \leq n \leq k+N-1\}$ in the following way:

$$\begin{aligned} \mathcal{P}_{k,N} = & \bigcup_{1 \leq d \leq d_0+1} \mathcal{P}'_{0,d} \cup \bigcup_{\substack{1 \leq m \leq t-1 \\ 1 \leq d \leq d_m}} \mathcal{P}'_{m,d} \cup \bigcup_{\substack{t \leq m \leq r-1 \\ 1 \leq d \leq d_m}} \mathcal{P}'_{m,d} \\ & \cup \bigcup_{0 \leq a \leq a_r-2} \mathcal{P}''_a \cup \bigcup_{\substack{0 \leq m \leq t-1 \\ 0 \leq x \leq a_m + \kappa_m - 1}} \mathcal{P}'''_{m,x} \cup \bigcup_{\substack{t \leq m \leq r-1 \\ 0 \leq x \leq a_m + \kappa_m - 1}} \mathcal{P}'''_{m,x}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}'_{m,d} &:= \{\mathbf{x}_{\ell b^r - d_{r-1} b^{r-1} - \dots - d_{m+1} b^{m+1} - d b^m + j} : 0 \leq j < b^m\}, \\ \mathcal{P}''_a &:= \{\mathbf{x}_{\ell b^r + a b^r + j} : 0 \leq j < b^r\}, \\ \mathcal{P}'''_{m,x} &:= \{\mathbf{x}_{(\ell+a_r-1)b^r + (\kappa_{r-1}+a_{r-1})b^{r-1} + \dots + (\kappa_{m+1}+a_{m+1})b^{m+1} + x b^m + j} : 0 \leq j < b^m\}. \end{aligned}$$

For $m \leq t-1$, we can bound the discrepancy of $\mathcal{P}'_{m,d}$ and $\mathcal{P}'''_{m,x}$, respectively, by the trivial bound 1. For $m \geq t$, the point sets $\mathcal{P}'_{m,d}$ and $\mathcal{P}'''_{m,x}$ are (t, m, s) -nets in base b , and the \mathcal{P}''_a are (t, r, s) -nets in base b . From the triangle inequality for the discrepancy we obtain

$$\begin{aligned} ND_N(\mathcal{P}_{k,N}) &\leq (d_0 + a_0 + \kappa_0 + 1)b^0 + \sum_{m=1}^{t-1} (d_m + a_m + \kappa_m)b^m \\ &\quad + \sum_{m=t}^{r-1} (d_m + a_m + \kappa_m)\Delta_b(t, m, s) + \max(a_r - 2, 0)\Delta_b(t, r, s) \\ &\leq (2b-1) + (2b-2) \left((t-1)b^t + \sum_{m=t}^{r-1} \Delta_b(t, m, s) \right) \\ &\quad + \max(b-3, 0)\Delta_b(t, r, s) \\ &\leq (2b-1) \left(tb^t + \sum_{m=t}^r \Delta_b(t, m, s) \right) \end{aligned}$$

and the result follows, since $r = \lfloor \log_b N \rfloor$. \square

Corollary 3. *Let $(\mathbf{x}_n)_{n \geq 0}$ be a (t, s) -sequence in base b . Then we have*

$$N\tilde{D}_N((\mathbf{x}_n)_{n \geq 0}) \ll_{s,b} b^t (\log N)^s.$$

Proof. The result follows from Theorem 5 together with the fact that

$$\Delta_b(t, m, s) \ll_{s,b} b^t m^{s-1}$$

for $m \geq t$ (see, for example, [6, 24]). □

6.2 Uniform discrepancy of Halton-sequences

Theorem 6. *Let $(\mathbf{x})_{n \geq 0}$ be a Halton-sequence in pairwise co-prime bases b_1, \dots, b_s . Then we have*

$$N\tilde{D}_N((\mathbf{x}_n)_{n \geq 0}) = \frac{1}{s!} \prod_{j=1}^s \left(\frac{\lfloor b_j/2 \rfloor \log N}{\log b_j} + s \right) + O((\log N)^{s-1}),$$

where the implied constant depends on b_1, \dots, b_s and s .

Proof. The result follows from an adaption of the proof of [6, Theorem 3.36]. Note that [6, Lemma 3.37] also holds true for $A(J, k, N, \mathcal{S}) := \#\{n \in \mathbb{N} : k \leq n < k + N \text{ and } \mathbf{x}_n \in J\}$ instead of $A(J, N, \mathcal{S}) := A(J, 0, N, \mathcal{S})$. The rest of the proof of [6, Theorem 3.36] remains unchanged. □

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